

# Two-Particle Dispersion in Model Velocity Fields

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We consider two-particle dispersion in a velocity field, where the relative two-point velocity scales according to  $v^2(r) \propto r^\alpha$  and the corresponding correlation time scales as  $\tau(r) \propto r^\beta$ , and fix  $\alpha = 2/3$ , as typical for turbulent flows. We show that two generic types of dispersion behavior arise: For  $\alpha/2 + \beta < 1$  the correlations in relative velocities decouple and the diffusion approximation holds. In the opposite case,  $\alpha/2 + \beta > 1$ , the relative motion is strongly correlated. The case of Kolmogorov flows corresponds to a marginal, nongeneric situation.

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Since the seminal work of Sir L.F.Richardson on particles' dispersion in atmospheric turbulence [1] a large amount of work has been done in order to understand the fundamentals of this process (see [2] and [3] for reviews). Based on empirical evidence, Richardson found out that the mean square distance  $R^2(t) = \langle r^2(t) \rangle$  between two particles dispersed by a turbulent flow grows proportionally to  $t^3$ . The works of Obukhov and Batchelor have shown that Richardson's law is closely related to the Kolmogorov-Obukhov scaling of the relative velocities in turbulent flows. Scaling arguments based on dimensional analysis allow then to understand the overall type of the behavior of  $R^2(t)$ , but a full theoretical picture of the dispersion process is still lacking [3].

The theoretical description of dispersion processes typically starts from models, in which one fixes the spacial statistics of the well developed turbulent flow (Kolmogorov-Obukhov energy spectrum), and discusses different types of temporal behavior for the flows [3]. Three situations have been considered in-detail so far. Here, the white-in time flows represent a toy model which allows for deep analytical insights [4-6]. In connection with "real" turbulence two other cases are widely discussed. One of them supposes that the temporal decorrelation of the particles' relative motion happens because the pair as a whole is moving with a mean velocity relative to an essentially frozen flow structure (as proposed by a Taylor hypothesis) [7,8]. Another premise connects this decorrelation with the death and birth of flow structures ("eddies"), whose lifetime is governed by Kolmogorov's universality assumption [8,9]. Both these situations are extremely awkward for theoretical analysis.

In the present letter we address the following question: What are the generic types of two-particle dispersion behavior in a velocity field whose statistical spatial structure is fixed (and similar to that of a turbulent flow), if its temporal correlation properties change. This question will be discussed in the framework of numerical simulations and scaling concepts. As we proceed to show, two generic types of behavior arise. Thus, the white-in-time flow and the Taylor-type situation belong to the classes of diffusive and ballistic behavior, respectively. The case of Kolmogorov temporal scaling represents a borderline

situation.

Let us consider modes of particles' separation in a velocity field whose two-time correlation function of relative velocities behaves as  $\langle \mathbf{v}(\mathbf{r}, t_1) \mathbf{v}(\mathbf{r}, t_2) \rangle \propto \langle v^2(r) \rangle g[(t_2 - t_1)/\tau(r)]$ , where  $\tau(r)$  is the distance-dependent correlation time. The  $g$ -function is defined so that  $g(0) = 1$  and  $\int_0^\infty g(s) ds = 1$ . The mean square relative velocity and the correlation time scale as

$$\langle v^2(r) \rangle \propto v_0^2 \left( \frac{r}{r_0} \right)^\alpha \quad (1)$$

and

$$\tau(r) \propto \tau_0 \left( \frac{r}{r_0} \right)^\beta. \quad (2)$$

One can visualize such a flow as being built up from several structures (plane waves, eddies, etc., see Ref.[3]), each of which is characterized by its own spatial scale and its scale-dependent correlation time. In well-developed turbulent flows one has  $v^2(r) \propto \epsilon^{2/3} r^{2/3}$ , where  $\epsilon$  is the energy dissipation rate, so that  $\alpha = 2/3$ . The white-in-time flow corresponds to  $\beta = 0$ . Kolmogorov scaling implies  $\beta = 2/3$  and Taylor's frozen-flow assumption leads to  $\beta = 1$ .

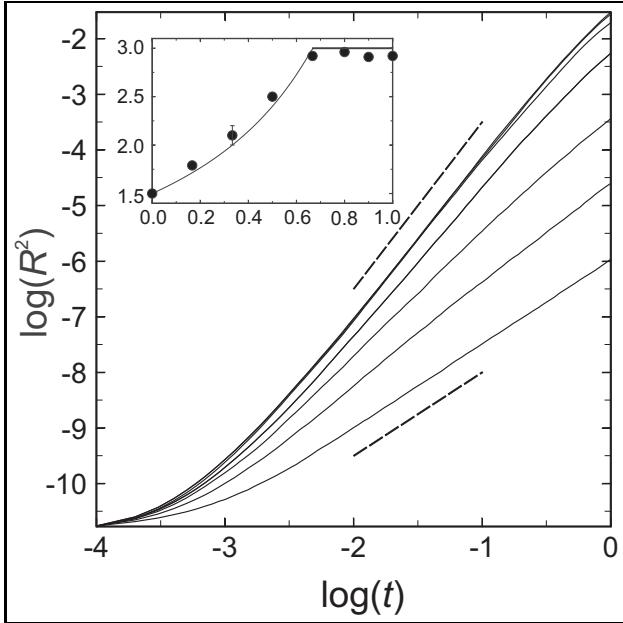
In our simulations we model the two-particle relative motion using the quasi-Lagrangian approach of Ref.[9]. Parallel to Ref.[9] we confine ourselves to a two-dimensional case, which is also of high experimental interest [12, 13]. The relative velocity  $\mathbf{v}(\mathbf{r}, t) = \nabla \times \eta(\mathbf{r}, t)$  is given by the quasi-Lagrangian stream function  $\eta$ . This function is built up from the contributions of radial octaves:

$$\eta(\mathbf{r}, t) = \sum_{i=1}^N k_i^{-(1+\alpha/2)} \eta_i(k_i \mathbf{r}, t), \quad (3)$$

where  $k_i = 2^i$ , and the flow function for one-octave contribution in polar coordinates  $(r, \theta)$  is given by  $\eta_i(k_i r, t) = F(k_i r) (A_i(t) + B_i(t) \cos(2\theta + \phi_i))$ . The radial part  $F(x)$  obeys  $F(x) = x^2(1-x)$  for  $0 \leq x \leq 1$  and  $F(x) = 0$  otherwise, and  $\phi_i$  are quenched random

phases. Moreover,  $A_i(t)$  and  $B_i(t)$  are independent Gaussian random processes with dispersions  $A^2 = B^2 = v_0^2$  and with correlation times  $\tau_i = 2^{-i\beta}\tau_0$ . At each time step these processes are generated according to  $X_i(t + \Delta t) = \sqrt{1 - (\Delta t/\tau_i)^2}X_i(t) + (\Delta t/\tau_i)v_0^2\zeta$ , where  $X$  is  $A$  or  $B$ , and  $\zeta$  is a Gaussian random variable with zero mean and unit variance. The values of  $\tau_0$  and the integration step  $\Delta t$  are to be chosen in such a way that  $\tau_N \gtrsim \Delta t$ . Typically, values of  $\Delta t \sim 10^{-4}$  are used. For the noncorrelated flow ( $\beta = 0$ ) the values of  $A_i(t)$  and  $B_i(t)$  are renewed at each integration step  $\Delta t$ . In the present simulations  $N = 16$  was used. The value  $v_0 = 1$  was employed in the majority of simulations reported here, so that only the use of a different  $v_0$  value will be explicitly stated in the following.

The values of  $R^2(t)$  obtained from 3000 realizations of the flow for several values of  $\beta \in [0, 1]$  are plotted on double logarithmic scales in Fig.1, where  $\tau_0 = 0.15$  is used. One can clearly see that for all  $\beta$  a scaling regime  $R^2(t) \propto t^\gamma$  appears. We note moreover that the curves for  $\beta = 0.67, 0.8, 0.9$  and  $1.0$  are almost indistinguishable within statistical errors. The values of  $\gamma$  as a function of  $\beta$  are presented in the insert, together with the theoretically predicted forms, vide infra.



**Fig. 1.** Mean square displacement  $R^2(t)$  plotted in double logarithmic scales. The four lower curves correspond to  $\beta = 0, 0.17, 0.33$  and  $0.5$  (from bottom to top). The dashed lines indicate the slopes  $1.5$  and  $3$ . The four upper curves for  $\beta = 0.67, 0.8, 0.9$  and  $1$  are hardly distinguishable within the statistical errors of the simulations. The insert shows the values of  $\gamma(\beta)$  determined by a least-squares-fit within the scaling region of each curve. The error bar shows the typical accuracy of all  $\gamma$ -values. The full lines give the theoretical predictions, Eq.(4) and Eq.(5).

The regimes of dispersion found in simulations can be explained within the framework put forward in Ref.[10].

The discussion starts by considering  $l(r) = v(r)\tau(r)$ , the mean free path of motion at the distance  $r$ . If this mean free path always stays small compared to  $r$ , the relative motion exhibits a diffusive behavior with a position-dependent diffusion coefficient,  $K(r) \propto l^2(r)/\tau(r) \propto r^{\alpha+\beta}$ . Taking as a scaling assumption  $r \propto \langle r^2(t) \rangle^{1/2} = R$ , one gets that the mean square separation  $R$  grows as  $R^2 \propto t^\gamma$  with

$$\gamma = \frac{2}{2 - (\alpha + \beta)}. \quad (4)$$

On the other hand, if  $l(r)$  is of the order of  $r$ , the mean separation follows from the integration of the ballistic equation of motion  $\frac{d}{dt}R = v(R) \propto R^{\alpha/2}$ , see Ref.[11]. Thus, in a flow where a considerable amount of flow lines of relative velocity are open, one gets  $R^2 \propto t^\gamma$ , with

$$\gamma = \frac{4}{2 - \alpha}. \quad (5)$$

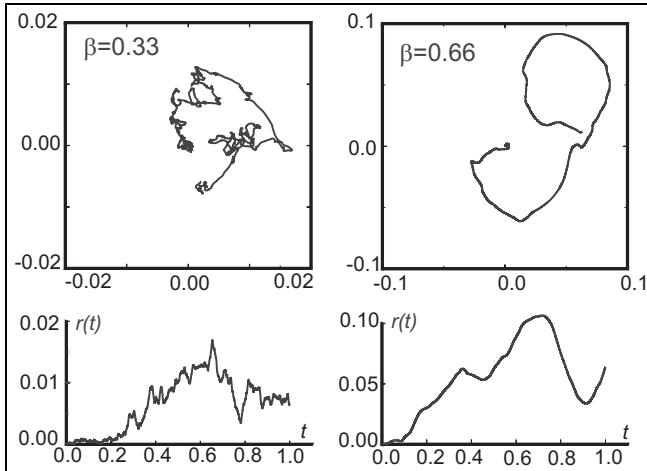
The occurrence of either regime is governed by the value of the (local) persistence parameter of the flow,

$$Ps(r) = l(r)/r = v(r)\tau(r)/r. \quad (6)$$

Small values of  $Ps$  correspond to erratic, diffusive motion, while large values of  $Ps$  imply that the motion is strongly persistent. The value of the persistence parameter scales with  $r$  as  $Ps(r) \propto r^{\alpha/2+\beta-1}$ . Since under particle's dispersion the mean interparticle distance grows continuously with time, the value of  $Ps$  decreases continuously for  $\alpha/2 + \beta < 1$ , so that the diffusive approximation is asymptotically exact. For  $\alpha/2 + \beta > 1$  the lifetimes of the structures grow so fast that the diffusive approximation does not hold. This situation is one observed in our simulations for  $\beta > 2/3$ . The strong ballistic component of motion implies that the velocities stay correlated over considerable time intervals. The results of Fig.1 confirm that  $\gamma(\beta)$  behaves accordingly to Eq.(4) for  $\beta < 2/3$  and Eq.(5) for  $\beta > 2/3$ . We note here that the parameters of the simulations presented in Fig.1 ( $v_0 = 1$ ,  $\tau_0 = 0.15$ ) were chosen in a way that allows to show all curves within the same time- and distance intervals. This leads to a somehow restricted scaling range and to slight overestimate of  $\gamma$ -values in the diffusive domain.

Strong differences between the diffusive and the ballistic regimes can be readily inferred when looking at typical trajectories of the motion, such as are plotted in Fig.2 for the cases  $\beta = 0.33$  and  $\beta = 0.67$ . The difference between the trajectories is evident both in the  $(x, y)$ -plots and in the  $r(t)$ -dependences. The curves for  $\beta = 0.33$  exhibit a random-walk-like, erratic behavior, while the curves for  $\beta = 0.67$  show long periods of laminar, directed motion. In order to quantitatively characterize the strength of the velocity correlations we calculate the backwards-in-time correlation function (BCF) of the radial velocities, as introduced in Ref.[12]. This function is defined as  $C_r(\tau) = \langle v_r(t - \tau)v_r(t) \rangle / \langle v_r^2(t) \rangle$  and shows,

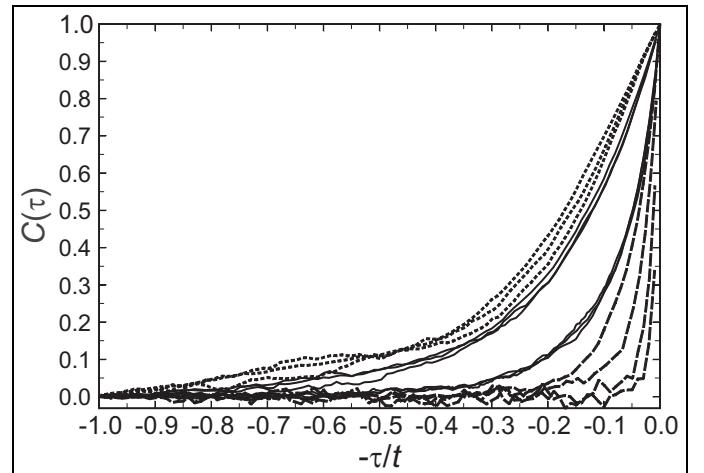
what part of its history is remembered by a particle in motion. The function is plotted in Fig.3 against the dimensionless parameter  $\vartheta = -\tau/t$ . The functions (obtained in  $10^4$  realizations each) are plotted for 4 different sets of parameters. Here the dashed lines correspond to  $\beta = 0.33$ , in the diffusive range, for  $t = 10^{-2}, 3 \cdot 10^{-2}, 10^{-1}$  and  $3 \cdot 10^{-1}$ . These BCF do not scale and are rather sharply peaked close to zero, thus indicating the loss of memory. The two sets of full lines indicate  $C_r(\tau)$  in Kolmogorov flows, for  $t = 10^{-2}, 3 \cdot 10^{-2}$ , and  $10^{-1}$ . The lower set corresponds to the value  $\tau_0 = 0.05$  and the upper set to the value  $\tau_0 = 0.15$ . In both cases the functions show scaling behavior. No considerable changes in the BCF's form occur when further increasing the value of  $\tau_0$  up to  $\tau_0 = 1$ , thus indicating that the data  $\tau_0 = 0.15$  correspond already to a strongly correlated regime. The form of these curves resembles closely the experimental findings of Ref. [12]. The BCFs for  $\beta = 1.0$  show an overall behavior very similar to the one in Kolmogorov's case. Note that as the time grows the curves for  $\beta = 1.0$  approach those for  $\beta = 2/3$  and probably tend to the same limit. The curves for  $\beta = 1.0$  and  $\tau_0 = 1$  (not shown) fall together with those in Kolmogorov's case with  $\tau_0 = 0.15$ .



**Fig. 2.** Typical trajectories in the diffusive regime ( $\beta = 0.33$ ) and in the Kolmogorov regime ( $\beta = 0.67$ ). The upper pictures show the trajectories in the  $(x, y)$ -plane, the lower ones represent the corresponding  $r(t)$ -behavior. Note that the scales of the right and of the left graphs differ by a factor of 5.

The similarity in the properties of dispersion processes in a Kolmogorov situation with larger  $Ps$  (larger  $\tau_0$ ) and in ballistic regime can be explained based on the behavior of the effective persistence parameter. In the diffusive regime we supposed that the correlation time of the particles' relative velocity scales in the same way as the Eulerian lifetime of the corresponding structures. On the other hand, in the ballistic regime,  $\beta > 1 - \alpha/2$ , the lifetimes of the structures grow so fast that no considerable decorrelation takes place during the time the particles sweep through the structure. The Lagrangian

decorrelation process is then connected not to Eulerian decorrelation, but to sweeping along open flow lines. The effective correlation time then scales according to  $\tau_s(r) \propto r/v(r) \propto t^{1-\alpha/2}$ , and the effective value of  $\beta$  stagnates at  $\beta = 1 - \alpha/2$ . Thus, all long-time correlated cases belong to the same universality class of strongly-correlated flows, as the Kolmogorov flows with large  $Ps$ , for which Eq.(4) and (5) coincide.



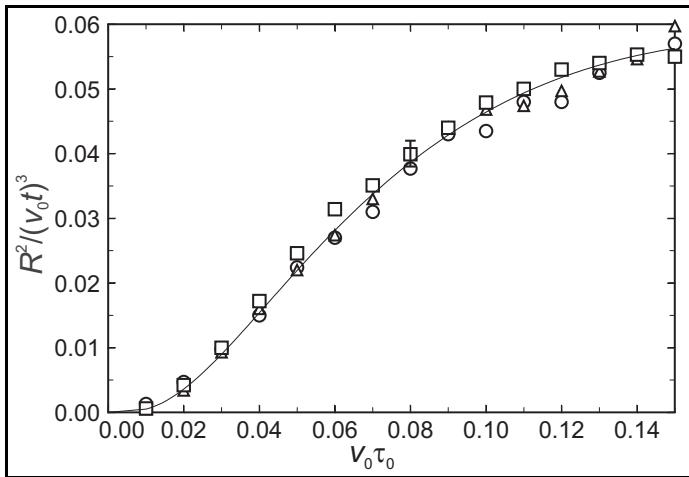
**Fig. 3.** The BCF of relative velocities as a function of the dimensionless time lag  $-\tau/t$ . The lower group of dashed lines corresponds to  $\beta = 0.33$  (the values of  $t$  are  $0.01, 0.03, 0.1$  and  $0.3$ , from top to bottom). The two groups of full curves corresponds to the Kolmogorov case (three curves for  $t = 0.01, 0.03, 0.1$  each). The dotted curves correspond to  $\beta = 1$  for the same values of time, see text for details.

For Kolmogorov flows the ballistic and the diffusive mechanisms lead to the same functional form of  $R^2(t)$ -dependence. The functional form of the dependence of  $R^2(t)$  on parameters of the flow is  $R^2 \propto (v_0^2 \tau_0 / r_0^{\alpha+\beta})^\gamma t^\gamma$  in the diffusive situation ( $Ps \ll 1$ ) and  $R^2 \propto (v_0 / r_0^{\alpha/2})^\gamma t^\gamma$  in the ballistic case ( $Ps \gg 1$ ). Assuming that  $Ps$  is the single relevant parameter governing the dispersion we are lead to the form  $R^2(t) \propto f(Ps) (v_0 / r_0^{\alpha/2})^\gamma t^\gamma$ , where  $f(Ps)$  is a universal function of  $Ps$ , which behaves as  $Ps^\gamma$  for  $Ps \ll 1$  and tends to a constant for  $Ps \gg 1$ . Thus, for a fixed spatial structure of the flow, the following scaling assumption is supposed to hold:

$$\frac{R^2(t)}{(v_0 t)^\gamma} = F(v_0 \tau_0), \quad (7)$$

which scaling can be checked in our case by plotting  $R^2(t)/(v_0 t)^3$  against  $v_0 \tau_0$ . The corresponding plot is given in Fig. 4, where we fix  $t = 0.1$ , and plot the results in three series of simulations. Each point corresponds to an average over  $5 \cdot 10^4$  runs. Here the squares correspond

to  $v_0 = 1$  and to the values of  $\tau_0$  ranging between 0.01 and 0.15, the triangles correspond to  $v_0 = 0.3$  and the  $\tau_0$  between 0.033 and 0.5, and the circles to  $\tau_0 = 0.1$  and to values of  $v_0$  between 0.1 and 1.5. The error bar indicates a typical statistical error as inferred from 5 similar series of  $5 \cdot 10^4$  runs each. The scaling proposed by Eq.(7) is well-obeyed by the results. Some points outside of the range of Fig.4 were also checked. Thus, for larger values of  $v_0\tau_0$  the values of  $R^2(t)/(v_0t)^3$  seem to stagnate. On the other hand, increasing  $v_0\tau_0$  to values larger than 0.3 (i.e. approaching the frozen flow regime) leads to a strong increase in fluctuations, making the results less reliable.



**Fig. 4.** The values of  $R^2(t)/(v_0t)^3$  plotted against  $v_0\tau_0$ . The dashed line is drawn as a guide to the eye.

Let us summarize our findings. Thus, we considered two-particle dispersion in a velocity field scaling according to  $v^2(r) \propto r^{2/3}$  and  $\tau(r) \propto r^\beta$ . We show that two generic types of behavior are possible: For  $\alpha/2 + \beta < 1$  the diffusion approximation holds and the increase in the interparticle distances is governed by the distance-dependent diffusion coefficient  $K(r) \propto r^{\alpha+\beta}$ . In the opposite case  $\alpha/2 + \beta > 1$  the relative velocities stay strongly correlated. The transition between the two regimes takes place exactly for the Kolmogorov flow, for which  $\alpha/2 + \beta = 1$ . In this case the properties of the dispersion process depend on the persistence parameter of the flow.

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